PHYS 705: Classical Mechanics

Remaining Weeks (five wks) for the Semester

HW#8 and #9: CT and Hamilton-Jacobi Eq (Nov 8 & 15)

HW#10: Small Oscillations (Nov 22)

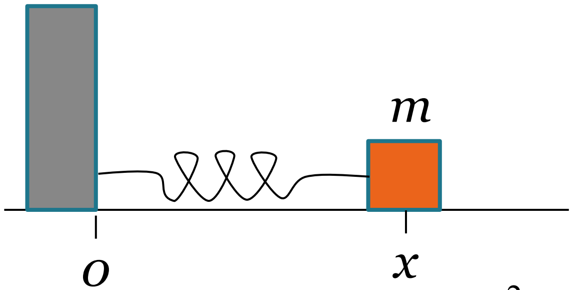
HW#11: Noninertial Reference Frame and Rigid Body Motion (Nov 29)

HW#12: Rigid Body Motion (more practice problems)

FINAL EXAM on Dec 6 (4:30-7:10p, Planetary 220)

Review from Previous Lecture on CT

Example: Harmonic Oscillator



$$f(x) = -kx \qquad U(x) = \frac{kx^2}{2}$$

$$L = T - U = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \rightarrow \quad \dot{x} = \frac{p}{m}$$

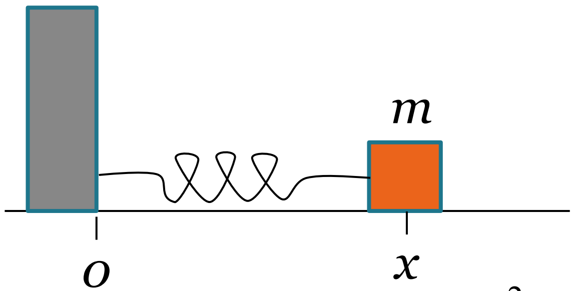
$$H = p\dot{x} - L \; = p\dot{x} - \frac{m\dot{x}^2}{2} + \frac{kx^2}{2}$$

$$= \frac{p^2}{m} - \frac{m}{2}\frac{p^2}{m^2} + \frac{kx^2}{2} \qquad (\text{in } x \,\&\, p)$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Define $\omega = \sqrt{k/m}$ or $\omega^2 m = k$ $\rightarrow$ $\boxed{H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)}$

Example: Harmonic Oscillator



$$f(x) = -kx \quad U(x) = \frac{kx^2}{2}$$

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)$$

Application of the Hamilton's Equations give,

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{2m^2\omega^2}{2m}x = -m\omega^2 x$$

Combining the two equations, we have the standard equation for SHO: $\ddot{x} + \omega^2 x = 0$

We then showed that we can find a canonical transformation from (x, p) to (X, P) such that X is *cyclic* so that P is constant.

Example: Harmonic Oscillator

Here is the following Canonical Transformation:

$$x = \sqrt{\frac{2P}{m\omega}} \sin X \qquad p = \sqrt{2m\omega P} \cos X$$

Under this CT, the transformed Hamiltonian becomes extremely simple:

$$K = \omega P$$

Applying the Hamilton's Equations gives,

$$\dot{P} = -\frac{\partial K}{\partial X} = 0 \quad (X \text{ is cyclic}) \qquad \dot{X} = \frac{\partial K}{\partial P} = \omega$$

$$\rightarrow \boxed{P = const} \qquad \rightarrow \boxed{X = \omega t + \alpha}$$

depends on IC

Example: Harmonic Oscillator

$$X(t) = \omega t + \alpha \qquad\qquad P = const$$

**Notice how simple the EOM are in the new transformed cyclic variables. In most application, the goal is to find a new set of canonical variables so that there are as many cyclic variables as possible.

Using the inverse transform : $x = \sqrt{\frac{2P}{m\omega}} \sin X$, we can write down the EOM in the original variable:

$$x = \sqrt{\frac{2P}{m\omega}} \sin(\omega t + \alpha) \qquad (P \text{ is a constant})$$

Poisson Bracket

For any two function $u(q,p)$ and $v(q,p)$ depending on q and p the Poisson Bracket is defined as:

$$[u,v]_{q,p} \equiv \left(\frac{\partial u}{\partial q_j}\right)\left(\frac{\partial v}{\partial p_j}\right) - \left(\frac{\partial u}{\partial p_j}\right)\left(\frac{\partial v}{\partial q_j}\right)$$

(E's sum rule for n dof)

PB is analogous to the Commutator in QM:

$$\frac{1}{i\hbar}[\![u,v]\!] \equiv \frac{1}{i\hbar}(uv - vu)$$

where u and v are two QM operators

“Symplectic” Approach & Poisson Bracket

Note that this symplectic structure for the canonical transformation can also be expressed elegantly using the matrix notation that we have introduced earlier :

Recall that the Hamilton Equations can be written in a matrix form,

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}}$$

with $\eta_j = q_j, \qquad \eta_{j+n} = p_j; \qquad j = 1, \cdots, n$

$$\left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_j = \frac{\partial H}{\partial q_j}, \qquad \left(\frac{\partial H}{\partial \boldsymbol{\eta}}\right)_{j+n} = \frac{\partial H}{\partial p_j}; \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

"Symplectic" Approach & Poisson Bracket

In terms of these matrix notation, we can also write the Poisson bracket as,

$$[u,v]_{\boldsymbol{\eta}} = \frac{\partial u}{\partial \boldsymbol{\eta}} \mathbf{J} \frac{\partial v}{\partial \boldsymbol{\eta}}$$

And if $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$ is a canonical transformation, then Fundamental Poisson Brackets can simply written as,

$$[\boldsymbol{\zeta},\boldsymbol{\zeta}]_{\boldsymbol{\eta}} = [\boldsymbol{\eta},\boldsymbol{\eta}]_{\zeta} = \mathbf{J}$$

Also if we have $M_{jk} = \frac{\partial \zeta_j}{\partial \eta_k}, \quad j,k = 1,\cdots,2n$ (Jacobian matrix) then, we have the following conditional check for a canonical transformation:

$$\boxed{\mathbf{MJM}^T = \mathbf{J}}$$

(this condition is typically easier to use than the direction condition for a CT)

Poisson Bracket & Dynamics

In terms of PB, we also have the following equation of motion for any dynamical quantity $u(t)$:

$$\dot{u} = \frac{du}{dt} = \left[u, H\right] + \frac{\partial u}{\partial t}$$

General Comments:

1. Applying the above equation with $u = H$, we have $\left[H, H\right] = 0$ and:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Poisson Bracket & Dynamics

General Comments:

2. If $u = q_j \ or \ p_j$, we get back the Hamilton's Equations:

$$\dot{q}_j = \left[q_j, H\right] = \frac{\partial H}{\partial p_j} \qquad \text{and} \qquad \dot{p}_j = \left[p_j, H\right] = -\frac{\partial H}{\partial q_j} \qquad \text{(hw)}$$

3. If u is a constant of motion, i.e., $\dfrac{du}{dt} = 0$

$$\dot{u} = \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \quad \Longrightarrow \quad [u, H] = -\frac{\partial u}{\partial t}$$

Specifically, for u explicitly not depends on time, i.e., $\dfrac{\partial u}{\partial t} = 0$,

$$\frac{du}{dt} = 0 \quad \Longleftrightarrow \quad [u, H] = 0$$

u and H "commute"!

Poisson Bracket & Dynamics

4. One can formally write down the time evolution of $u(t)$ as a series solution in terms of the Poisson brackets evaluated at $t = 0$!

$$u(t) = u(0) + t\left[u, H\right]_0 + \frac{t^2}{2!}\left[\left[u, H\right], H\right]_0 + \cdots$$

The Hamiltonian is the generator of the system's motion in time !

⇨ The above Taylor's expansion can be written as an "operator" eq:

$$u(t) = e^{\hat{H}t} u(0)$$

where $\hat{H} = \left[\ \ , H\right]\Big|_0$

⟺

$$\left|u(t)\right\rangle = e^{iHt/\hbar}\left|u(0)\right\rangle$$

(QM propagator)

This has a direct correspondence to the QM interpretation of H.

Hamilton-Jacobi Equation

The HJ eq results when we enforce Q and P to be constants in time and the transformed Hamiltonian K need to be identically zero.

If that is the case, the equations of motion will be,

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$$

$$Q_i = \beta_i$$

$$P_i = \alpha_i$$

Recall that we have the new and old Hamiltonian, K and H, relating through the generating function F_2 (using type 2) by:

$$K = H + \frac{\partial F_2}{\partial t}$$

Hamilton-Jacobi Equation

Then, one can formally rewrite the equation (with $K = 0$) as:

$$H\left(q_1,\cdots,q_n;\frac{\partial F_2}{\partial q_1},\cdots,\frac{\partial F_2}{\partial q_n};t\right)+\frac{\partial F_2}{\partial t}=0$$

This is known as the Hamilton-Jacobi Equation.

Notes: -Since $F_2\left(q,P,t\right)$ and $P_i=\alpha_i$ are constants, the HJ equation constitutes a partial differential equation of (n+1) independent variables: $\left(q_1,\cdots,q_n,t\right)$

- It is customary to denote the solution F_2 by S and called it the Hamilton's Principal Function.

Hamilton-Jacobi Equation

Writing the Hamilton Principal Function out explicitly,

$$F_2 \equiv S = S\left(q_1, \cdots, q_n, \alpha_1, \cdots, \alpha_n; t\right)$$

in terms of its (n+1) independent variables $\left(q_1, \cdots, q_n; t\right)$ and constants $P_i = \alpha_i$

After we get an exp for S from the HJ Eq, we can solve for p_i(t) and q_i(t) using the following two partial differential eqs:

$$p_i = \frac{\partial S\left(q, \alpha, t\right)}{\partial q_i} \quad (T1)$$

$$Q_i = \beta_i = \frac{\partial S\left(q, \alpha, t\right)}{\partial \alpha_i} \quad (T2)$$

Hamilton-Jacobi Equation

1. Using (T1) and *initial* conditions at time t_0 , one can solve for the *n* unknown constants α_i in terms of the *initial conditions*, i.e.,

$$p_i\left(t_0\right)=\left.\frac{\partial S\left(q,\alpha,t\right)}{\partial q_i}\right|_{q=q_0,t=t_0}$$

$$\Rightarrow \qquad \alpha_i=\alpha_i\left(q_{0,}p_0,t_0\right)$$

2. Then, by using (T2) again at time t_0 , we obtain the other *n* constants of motion β_i

$$Q_i=\beta_i=\left.\frac{\partial S\left(q,\alpha,t\right)}{\partial \alpha_i}\right|_{q=q_0,t=t_0}$$

Hamilton-Jacobi Equation

3. With all 2n constants of motion α_i, β_i solved, we can now again use Eq. (T2) again to solve for q_i in terms of the α_i, β_i at a later time *t*.

$$\boxed{q_i = q_i(\alpha, \beta, t)} \qquad \left(\beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}\right)$$

4. With α_i, β_i, and q_i known, we can use Eq. (T1) again to solve for p_i in terms of α_i, β_i at a later time *t*.

$$\boxed{p_i = p_i(\alpha, \beta, t)} \qquad \left(p_i = \frac{\partial S(q(\alpha, \beta, t), \alpha, t)}{\partial q_i}\right)$$

➡ The two boxed equations constitute the desired complete solutions of the Hamilton equations of motion.

Hamilton’s Characteristic Function

Let consider the case when the Hamiltonian is constant in time, i.e.,

$$H\left(q_i, p_i\right) = \alpha_1$$

Now, let also consider a canonical transformation under which the new momenta are all constants of the motion,

$$P_i = \alpha_i$$

(the transformed Q_i are not restricted a priori.)

AND H is the new canonical momentum α_1, $\left(H\left(q_i, p_i\right) = \alpha_1\right)$

Then, we seek to determine the time-independent generating function $W(q_i, P_i)$ (Type-2) producing the desired CT.

Hamilton's Characteristic Function

Similar to the development of the Hamilton's Principal Function, since $W(q,P)$ is Type-2, the corresponding equations of transformation are

$$p_i = \frac{\partial W(q,\alpha)}{\partial q_i} \quad (T1)$$

$$Q_i = \frac{\partial W(q,\alpha)}{\partial \alpha_i} \quad (T2)$$

(Note: the indices inside $W(q, P)$ are being suppressed.)

Now, since $W(q,P)$ is time-independent, $\dfrac{\partial W(q,\alpha)}{\partial t} = 0$ and we have

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) + \cancel{\frac{\partial W}{\partial t}} = K = \alpha_1$$

Hamilton's Characteristic Function

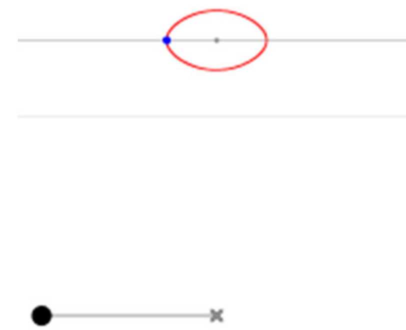
$W(q,P)$ is called the Hamilton's Characteristic Function and

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) - \alpha_1 = 0$$

is the partial differential equation (Hamilton-Jacobi Equation) for W.

Here, we have n independent constants α_i (with $\alpha_1 = H$) in determining this partial diff. eq.

Action-Angle Variables in 1dof



$$q(t+T) = q(t)$$

- Often time, for a system which oscillates in time, we might not be interested in the details about the EOM but we just want information about the *frequencies* of its oscillations.

- The H-J procedure in terms of the Hamilton Characteristic Function can be a powerful method in doing that.

- To get a sense on the power of the technique, we will examine the simple case when we have only one degree of freedom.

- We continue to assume a conservative system with $H = \alpha_1$ being a constant

Action-Angle Variables in 1dof

- Now, we introduce a new variable

$$J = \oint p\, dq$$

called the **Action Variable**, where the path integral is taken over one full cycle of the periodic motion.

- Now, instead of requiring our new momenta P to be α_1 , we requires

$$P = J \quad \text{(another constant instead of } \alpha_1 \text{)}$$

- Then, our Hamilton Characteristic Function can be written in term of J

$$W = W\left(q, J\right)$$

Action-Angle Variables in 1dof

- Since the generating function $W(q, J)$ is time independent, the Hamiltonian in the transformed coordinate K equals to H so that

$$\alpha_1 \equiv H = H(J) = K(J)$$

$$\vdots$$

- The frequency of the periodic oscillation associated with q, $\nu(J) = 1/T$ can be directly evaluated thru

$$\nu(J) = \frac{\partial K(J)}{\partial J}$$

without finding the complete EOM